Reply to "Commentary on "Total Hadronic Cross Section Data and the Froissart-Martin Bound", by Fagundes, Menon and Silva"

D.A. Fagundes, M.J. Menon, P.V.R.G. Silva
Universidade Estadual de Campinas - UNICAMP
Instituto de Física Gleb Wataghin
13083-859 Campinas, SP, Brazil
fagundes@ifi.unicamp.br,
menon@ifi.unicamp.br, precchia@ifi.unicamp.br

A reply to the above mentioned commentary by M.M. Block and F. Halzen on our quoted work is presented. We answer to each point raised by these authors and argument that our data reductions, strategies and methodology are adequate to the nonlinear-fit-problem in focus. In order to exemplify some arguments, additional information from our subsequent analysis is referred to. A brief commentary on the recent results by Block and Halzen is also presented. We understand that this reply gives support to the results and conclusions presented in our quoted work.

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I. INTRODUCTION

In a recent communication M.M. Block and F. Halzen [1] (hereafter referred to as BH) have presented some critical comments on our analysis [2] (hereafter FMS). Some points raised by these authors have already been addressed and discussed in our subsequent work [3] (hereafter FMS2), available as arXiv since August 16, 2012. In this note we reply to the BH criticisms directed to FMS, recalling also some aspects of interest presented in FMS2.

First, to facilitate the discussion and the reference to each part of the BH commentaries, some explanations on the organization of this reply are in order. Excluding the acknowledgments and references, BH arXiv comprises four pages and the effective criticisms to FMS correspond to approximately one page. All the rest of the manuscript (three pages) largely overlap with their previous work [4] (as advised in the "arXiv admin note" [1]). We shall focus on this 25 % material, in our section II. Although not characterized as criticisms, the rest of the BH reproduces their former work on the subject, as a kind of lesson to be learned. In this respect, a discussion on some aspects of the FMS analysis and a brief commentary on the BH former work are presented in section III. Our conclusions and final remarks are the contents of section IV.

II. REPLY TO THE CRITICISMS

The content of the criticisms to FMS, presented in BH, can be divided in three blocks, one block referring to the ρ information (page 1 in BH), another block referring to statistical probabilities (page 2) and the last one to predictions at 7 TeV (page 2). In what follows, each block will be treated as a subsection, in which we first situate and summarize the commentary, or quote it explicitly, and then present our reply.

A. On the ρ Information

- Commentary

The first effective criticism appears in page 1, fourth paragraph of the section *Introduction*. It concerns the fact that in FMS the ρ information was not used in global fits with the total cross section data. According to them, "a statement alluded to (but *not* carried out) in Appendix...". They also add: "... in their Appendix, they give a rather cumbersome evaluation using their Variant 3 model, to *separately evaluate* ρ...."

- Our reply

In FMS, the analysis has been based only on the σ_{tot} data (without the inclusion of the ρ information) for the six reasons explained there, which we consider as six facts. However, addressing the comments by one of the *three referees* in the submission to the Braz. J. Phys., we have included Appendix A in a revised version. In this appendix we have shown that, even in the case of the largest values of the exponent γ (Method 1, V3 and Method 2, V5), the *predictions* for $\rho(s)$ are in agreement with the experimental information.

To connect $\sigma_{tot}(s)$ and $\rho(s)$ in an analytical way, we have used singly-subtracted derivative dispersion relations in the operational form introduced by Kang and Nicolescu [5] (also discussed in [6]). In particular we have obtained an *extended* analytical result for the case of γ as a *real* parameter (equations A7 and A8 in Appendix A of FMS).

In respect to the effect of the ρ information in data reductions, we have stated at the end of Appendix A (the references that follows concern FMS): "Finally, we recall that in simultaneous fit to σ_{tot} and ρ the subtraction constant affects both the low- and high-energy regions [47,48]. That is a consequence of the strong correlation among the subtraction constant and all the other physical free fit parameters. We plan to discuss this consequence and other aspects of the fit procedures in a forthcoming paper." Also, in the last paragraph of our conclusions (third sentence) we added: "These are aspects that we expect to consider in a future work, since they may provide information that is complementary to the results here presented."

In fact, in the FMS2 we have extended the FMS analysis in several aspects as referred to and outlined in the introduction of FMS2. In special, not only individual but also novel simultaneous fits to total cross section and ρ information have been developed, leading to solutions with γ greater than 2, *despite the constraint involved.* The origin and role of the subtraction constant have also been discussed in some detail.

Therefore, we see no sense in depreciating statements like "alluded but not carried out" or "they give a rather cumbersome evaluation".

B. Statistical Probabilities

The second criticism in BH appears in page 2, section *Statistical Probabilities* and involves five paragraphs, four with criticisms (left column) and the final one with the conclusions (right column). The main point here concerns the use of the integrated probability $P(\chi_0^2; \nu)$ to punctually analyze the reliability of the fit results. We shall treat each paragraph separately and in sequence. However, before that, some aspects on our fit procedures and on statistical analysis demand a few comments for further reference.

- Introductory remarks

- *On the fit procedures*

The parametrization for the total cross section used in FMS, introduced by Amaldi *et al.* [7], reads

$$\sigma_{tot}(s) = a_1 \left[\frac{s}{s_l}\right]^{-b_1} + \tau\, a_2 \left[\frac{s}{s_l}\right]^{-b_2} + \alpha + \beta \ln^\gamma\left(\frac{s}{s_h}\right), \qquad (1)$$

where $\tau = $ -1 (+1) for pp ($\bar{p}p$) scattering and $s_l = 1$ GeV^2 is fixed. The dependence is linear in four parameters (a_1, a_2, α and β) and nonlinear in the other four (b_1, b_2, γ and s_h). As stated by Bevington and Robinson [8] (section 8.2 Searching Parameter Space) and also quoted recently in [9],

> "Fitting nonlinear functions to data samples sometimes seems to be more an art than a science."

Nonlinear data reductions are not a trivial task. They demand a methodology for the choice of the initial (feedback) values of the free parameters. Our strategy has been to test a *grid* of different (physical) feedback values for the free fit parameters so as to check the stability of the results, as shortly recalled in what follows.

In both cases, FMS and FMS2, we have considered as feedback the results already found by the Particle Data Group (PDG), which uses the standard COMPETE Collaboration highest-rank parametrization ($\gamma = 2$, a fixed parameter) [10, 11]. In FMS we used the 2010 PDG edition [12] and in FMS2 the updated result from the 2012 PDG edition [13]. Although applied to only a *subset* of the dataset analyzed in PDG, we understand that with these "conservative choices" ($\gamma = 2$), we start with reasonable stable solutions (already found by the PDG). With this input we are then able to investigate possible *departures from this solution* in the case of γ as a free parameter (including or not the TOTEM datum [14], as done in FMS and including this point in all fits, as done in FMS2). *In addition*, in order to investigate the effect of the feedback values in the fit results, we have considered another distinct choices in both FMS and FMS2 and 3 different versions in each case of FMS (referred to as six variants and a direct fit).

As explicitly quoted in FMS and FMS2, our data reductions have been carried out with the objects of the class TMinuit of the ROOT Framework [15]. The statistical interpretations of the fit results, as well as aspects related to the error matrix, correlation matrix and analytical error propagation, have been based on the Bevington-Robinson book [8].

For further reference, we recall that the error (or covariance) matrix provides the variances (diagonal) and covariances (out of diagonal) associated with each free fit parameter. The symmetric correlation matrix gives a measure of the correlation between each pair of free parameter through a coefficient with numerical limits ± 1 (full correlation) and 0 (no correlation).

In the minimization program [16–18] a Confidence Level of one standard deviation was adopted in all fits (UP = 1). In each test of fit, successive running of the Migrad have been considered (up to 200 calls in FMS and up to 2,000 calls in FMS2), until full convergence has been reached, with the smallest FCN (χ^2) and EDM (Estimated Distance to Minimum), specifically EDM $< 10^{-4}$ (adequate for the one sigma CL). In addition, the error in the parameters should not exceed the central value.

Among *several different tests*, we have *selected*, under the above criteria, the seven variants presented in FMS. These variants are related to two different choices for the input values for all fits, denoted Method 1 and Method 2 in FMS. For further reference and clarity, we summarize and situate below the structure of the grid considered in FMS (V stands for variant and DF for direct fit, also a variant).

Method 1. Initial feedback values from PDG 2010 (Table 1, second column):

$$\left.\begin{array}{l} \sqrt{s}_{max} = 1.8 \text{ TeV Ensemble (Table 1)} \\[1em] \sqrt{s}_{max} = 7 \text{ TeV Ensemble (Table 2)} \end{array}\right\} \text{ variants DF, V1, V2, V3}$$

Method 2. Initial feedback values distinct from Method 1 (Table 3, second column):

$$\left.\begin{array}{l} \sqrt{s}_{max} = 1.8 \text{ TeV Ensemble (Table 3)} \\[1em] \sqrt{s}_{max} = 7 \text{ TeV Ensemble (Table 4)} \end{array}\right\} \text{ variants V4, V5, V6.}$$

In the six variants (including the DF) we have investigated the effects of fixing or not the three fundamental parameters directly related with the energy dependence, namely b_1, b_2 and γ. For the Reggeon intercepts we have tested either *ad hoc* fixed values $1/2$ or fixed (central) values from spectroscopic data (Chew-Frautschi plots).

We stress that these results constitute final solutions, selected under the above mentioned criteria. Therefore our strategy in FMS (and FMS2) did not involve extremely detailed use of different routines to possibly reach an absolute minimum. Our point has been to investigate a *grid* of reasonable physical choices for feedbacks and variants for two ensembles, obtaining solutions through standard running of the MINUIT. Despite of these possible limitations, we have found several reasonable consistent solutions with γ greater than 2 and that has been the only essential point raised in FMS (and developed also in FMS2).

- *On statistics*

In FMS and FMS2, following the PDG procedure, the dataset include statistical and *systematic errors*, added in quadrature. In our opinion, the inclusion of systematic errors puts certain limits in a *full* statistical interpretation of the fit results.

In fact, the χ^2 test for goodness of fit is based on the assumption of a *Gaussian error distribution* [8]. Although statistical uncertainties are considered to follow this distribution, that, certainly, is not the case for systematic uncertainties, which are equally probable quantities. Therefore, we understand that a *full* statistical interpretation of data reductions including systematic uncertainties has a somewhat limited validity, specially in what concerns integrated probability (due to the inclusion of equally probable quantities).

In the FMS and FMS2, the corresponding DOF (ν) and χ^2/ν for each fit has been displayed only to shown that they constitute reasonable (acceptable) statistical results. The condition of reduced χ^2 closest to 1.0 has been one of the criteria used to select a given result, but not the only one. Attempts to a full statistical interpretation of the results, mainly in terms of integrated probabilities, may lead to questionable conclusions, as discussed in what follows.

At last, it is important to note that the focus in FMS does not concern comparison among models (or variants), but between two ensembles, with or without the TOTEM datum. Also, when we refer to *statistically consistent results* for $\gamma = 2$ or γ above 2, we mean that *the corresponding numerical result for γ is consistent within their uncertainty*.

Let us now treat each paragraph from section *Statistical Probabilities* in BH. We shall adopt here their notation (lower index 1 for the $\sqrt{s}_{max} = 1.8$ TeV ensemble and lower index 2 for the $\sqrt{s}_{max} = 7$ TeV ensemble).

• First Paragraph

- Commentary

BH discuss our results in Table 1 of FMS ($\sqrt{s}_{max} = 1.8$ TeV), comparing the data reductions that can be summarized as follows:

DF$_1$: $\nu = 156$, $\chi^2/\nu = 0.931$, $P_{DF_1} = 0.721$,   case of $\gamma = 2$ (fixed)

V1: $\nu = 155$, $\chi^2/\nu = 0.937$, $P_{V1} = 0.701$,   case of γ free

According to Block and Halzen: "... we get the somewhat strange result that FMS have a ...*better, somewhat more reliable* fit when they fix the value of γ at 2,..., than they allow it to float, suggesting perhaps that the *true* minimum χ^2 was not achieved in their minimization process." They state in the last sentence: "In any event, FMS concluded that the value $\gamma = 2$ was correct for the energy interval $5 \leq \sqrt{s} \leq 1,800$ GeV."

- Our reply:

The V1 result has been obtained from DF$_1$ as feedback. Based on Bevington and Robinson [8] (Chapter 11) and in our above comments on statistics, for $\nu \sim 155/156$ we do not think that $P_{DF_1} \sim 0.72 > P_{V1} \sim 0.70$ strictly implies in a better fit, namely that DF$_1$ could be "more reliable" than V1. In fact, let us compare the values of the free parameters in both fits in Table 1 of FMS (third and fourth columns). Note that the central values of the parameters are identical up to 3 figures, except for β (0.264 and 0.263) and s_h (12.0 and 12.2). All the parameters in both fits are consistent within their errors, leading to the conclusion that both results are, effectively, equivalent (the corresponding curves in Figures 1 and 2 of FMS overlap).

We understand that in this nonlinear fit, based on the same dataset ($\sqrt{s}_{max} = 1.8$ TeV), to let free *only one parameter* (γ) does not allow the punctual statistical interpretation by Block and Halzen (we shall return to this point in the discussion on the fourth paragraph). The very small differences in the central values of the parameters, and those in the errors, are associated with the *correlations among all the fit parameters*, resulting, in this particular case, in a small decrease of the probability when γ is let free. The correlation coefficients associated with both fits are displayed in Table I: DF_1 above the diagonal of the table and V1 below the diagonal (we shall return to this point in what follows). Perhaps there can be some small differences in reaching a true minimum, but, in our opinion, that does not invalidate our results and interpretation.

TABLE I: Correlation coefficients from the (symmetric) correlation matrices [8, 15] associated with DF_1 and $V1_1$ results. The off-diagonal coefficients from DF_1 are displayed above the diagonal of the table (not filled) and those from V_1, below that diagonal.

		DF$_1$							
		a_1	b_1	a_2	b_2	s_h	α	β	γ
	a_1		0.504	0.259	0.238	0.089	0.221	-0.090	-
	b_1	0.754		0.079	0.073	0.897	0.953	0.772	-
	a_2	0.187	0.034		0.989	0.024	0.021	0.024	-
V1$_1$	b_2	0.163	0.031	0.981		0.024	0.020	0.026	-
	s_h	-0.363	-0.059	0.072	0.071		0.987	0.966	-
	α	-0.181	0.462	-0.059	-0.056	0.629		0.914	-
	β	0.028	-0.138	0.017	0.013	0.538	-0.116		-
	γ	-0.174	0.020	0.029	0.031	-0.234	0.237	-0.912	

Concerning the last sentence in the paragraph (quoted above), it is obvious that we were not led to that conclusion based only in this particular case (DF_1 and V1), but in all the methods and variants displayed in Tables I and III, as explained along the text in FMS (and summarized in the beginning of section IV).

This first example already indicates the limits of a full statistical interpretation in terms of the integrated probability. This interpretation, however, permeates all the paragraphs in BH, to be discussed next.

- Second Paragraph

Here they compare DF (γ fixed) and V1 (γ free) from Table 1 of FMS ($\sqrt{s}_{max} = 1.8$ TeV) and Table 2 ($\sqrt{s}_{max} = 7$ TeV). Let us treat the two cases (DF and V1) separately.

- Commentary

In the case of DF (γ fixed), summarizing the results,

DF$_1$: $\nu = 156$, $\chi^2/\nu = 0.931$, $P_{DF_1} = 0.721$, $\sqrt{s}_{max} = 1.8$ TeV

DF$_2$: $\nu = 157$, $\chi^2/\nu = 0.930$, $P_{DF_2} = 0.725$, $\sqrt{s}_{max} = 7$ TeV

They conclude:

"Thus, if $\gamma = 2$ is satisfactory for the low energy data, it appears to be exactly the same level of confidence when we include the Totem point."

- Our reply:

First, it is important to stress that DF$_2$ has been obtained with DF$_1$ as feedback. Since $\gamma = 2$ is fixed and only the TOTEM point has been added, it is not expected a drastic change in the fit results in terms of χ^2/ν (and possibly in terms of integrated probability) as is the case. However, note that the *high energy parameters* (α, β and s_h) are not identical up to 2 figures. In particular, the value of s_h in DF$_1$ is two times the value in DF$_2$. As a consequence, from Fig. 1 of FMS, the high-precision TOTEM result is not adequately described in both cases: the curve from DF$_1$ ($\sqrt{s}_{max}$ = 1.8 TeV) lies below the lower error bar and even DF$_2$ ($\sqrt{s}_{max} = 7$ TeV) lies only through the lower error bar. At this point we agree that a solid conclusion demands the evaluation of the uncertainties in the curves from error propagation (as we have done in FMS2). However, we do not think this lack

here invalidates our arguments (we shall return to this point in the next section). Therefore, we can not agree with the above quoted conclusion by Block and Halzen.

- Commentary

In the case of V1 (γ free), summarizing the results,

V1$_1$: $\nu = 155$, $\chi^2/\nu = 0.937$, $P_{V1_1} = 0.701$, $\sqrt{s}_{max} = 1.8$ TeV

V1$_2$: $\nu = 156$, $\chi^2/\nu = 0.935$, $P_{V1_2} = 0.709$, $\sqrt{s}_{max} = 7$ TeV

Here, however, BH compare V1$_2$ with DF$_2$, concluding that, once $P_{V1_2} = 0.709 < P_{DF_2} = 0.725$, "this result is somewhat strange, since the logarithmic power γ was let adjustable in the V1$_2$."

- Our reply:

The comparison is not adequate. As explained in the text and summarized in the legend of Tables 1 and 2, each result in Table II has been obtained using as feedback the corresponding results in Table 1. Specifically, DF$_2$ (Table 2) uses DF$_1$ (Table 1) as initial values in the data reductions and the same is true for V1$_2$ (Table 2) and V1$_1$ (Table 1) and so on.

Therefore, V1$_2$ and DF$_2$ have been obtained from different feedback values, which may imply in different regions of the χ^2 minimum (and that is the essence of our strategy, as already discussed in the introductory remarks to this section).

Here, it may be interesting to note that, if we accept the BH confidence on $P(\chi_0^2; \nu)$ and focus on V1 then

$$P_{V1_2} \sim 0.71 > P_{V1_1} \sim 0.70$$

would imply in a solution with $\gamma_{V1_2} \sim 2.10 \pm 0.03$ ($\sqrt{s}_{max} = 7$ TeV) more reliable than a solution with $\gamma_{V1_1} \sim 2.00 \pm 0.03$ ($\sqrt{s}_{max} = 1.8$ TeV), corroborating therefore our conclusion. That all seems a vicious circle.

Anyway, from these arguments we can not agree with the above quoted conclusion by Block and Halzen: we see nothing "somewhat strange" in our results.

• Third Paragraph

- Commentary

Here they insist in the punctual statistical interpretation, quoting now two fit results, V4 and V5, with Method 2 (different initial values from Method 1).

- Our reply:

In this case, they correctly compare two results associated with the same feedback values (first column in Table 3). However and once more, in our opinion,

$$P_{V4_1} \sim 0.62 > P_{V5_1} \sim 0.60$$

for $\gamma = 2$ and γ free, respectively, does not imply in a significant difference. There is only *one additional free parameter*, the same dataset and the resulting

$$\chi^2_{V4_1} = 152.154 \text{ and } \chi^2_{V5_1} = 152.133$$

reinforces our opinion. We attribute the differences in the values of the parameters to the correlations among them (5 free parameters in V4 and 6 free parameters in V5). Also, and once more, perhaps there can be some small differences in reaching a true minimum, but that does not invalidate our results and interpretation.

• Fourth Paragraph

- Commentary

"To illustrate this anomaly, we recall to the reader that the difference between the V4$_1$ and the Direct Fit (DF$_1$) models was that in the V4 model, the Regge powers b_1 and b_2 were fixed at 1/2, whereas they were allowed to vary in the Direct Fit model, *raising* the probability from $P_{V4} = 0.616$ to $P_{V6} = 0.721$, as expected - the *exact opposite* of the V4 to V5 effect, where the logarithmic power γ was varied from 2. Clearly, we question their minimization program, or their use of it."

- Our reply:

Here, there seems to be some errors (misprints?) along the text, leading to a complete misleading paragraph. That was the reason why we have quoted here the complete paragraph.

If we understood, they are attempting to compare, once more, different methods, namely DF (Method 1 - initial values from the COMPETE - PDG 2010, given in Table 1) with V4 (Method 2 - different initial values, given in Table 3). They refer to $P_{V4_1} = 0.616$ and $P_{V6_1} = 0.721$, but in the last case it should be P_{DF_1}, which reads 0.721 and not P_{V6} that reads 0.701. The paragraph seems to us rather anomalous.

It just seems strange these possible errors concerning V4 and V6 in Table 3 because these variants constitute very illustrative results. They are interesting examples because V6 has been obtained with V4 as feedback. The V4 has 3 fixed parameters, b_1, b_2 (low energies) and γ (high energies) and in V6 these *three* parameters are let free. They can be summarized as before:

V4: $\quad \nu = 158, \quad \chi^2/\nu = 0.963, \quad P_{V4} = \mathbf{0.616} \quad (b_1, b_2, \gamma \text{ fixed})$

V6: $\quad \nu = 155, \quad \chi^2/\nu = 0.937, \quad P_{V6} = \mathbf{0.701} \quad (b_1, b_2, \gamma \text{ free})$

Here, in this nonlinear fit, we have *three fit parameters floating* resulting in an effective rising of the integrated probability. The corresponding correlation coefficients for both fits are given in Table II: V4 above the diagonal of the Table and V6 below that diagonal.

TABLE II: Same as Table I for the V4 result (above the diagonal) the the V6 result (below the diagonal).

		a_1	b_1	a_2	b_2	s_h	α	β	γ
						V4$_1$			
	a_1		-	0.156	-	-0.965	-0.991	-0.889	-
	b_1	0.744		-	-	-	-	-	-
	a_2	0.181	0.034		-	-0.087	-0.101	-0.065	-
V6$_1$	b_2	0.158	0.031	0.981		-	-	-	-
	s_h	-0.381	-0.051	0.074	0.072		0.988	0.972	-
	α	-0.193	0.471	-0.058	-0.055	0.632		0.929	-
	β	0.003	-0.157	0.023	0.018	0.565	-0.102		-
	γ	-0.153	0.050	0.022	0.024	-0.282	0.228	-0.920	

• Fifth Paragraph

- Commentary
Block and Halzen conclude there is no statistical evidence for $\gamma > 2$.

- Our reply:
Based on all the results presented in Tables 1 to 4 of FMS, we conclude that, in what concerns fits with the used parametrization and once the TOTEM data is included in the dataset, we obtain statistical solutions consistent with γ greater than 2.

• Final Comment

Concerning the noticed "anomalous" decrease of the integrated probability when one parameter (γ) is let free, we add the comment that follows. The parameters a_1, b_1, a_2 and b_2 are related to the low and intermediate energy region ($5 \text{ GeV} \leq \sqrt{s} \lesssim 100 \text{ GeV}$), where the number of data points is much larger then that at higher energies (where the parameters α, β, γ and s_h play the central role). As a consequence of the correlation among all parameters, when γ is let free, the low-energy parameters may also be affected, leading, in some cases, to a less reliable fit result due to the large number of data points at the lower energies. Therefore, we understand that this possible "anomalous" effect may not be connected with the position of a true minimum, but a consequence of three aspects: (1) the particular analytical structure of the parametrization; (2) the characteristics of the dataset (in terms of the energy); (3) the correlation among all the free fit parameters. In Tables I and II we illustrate these correlations in the case of DF$_1$ and V1$_1$ and also V4$_1$ and V6$_1$, respectively.

C. Prediction at the 7 TeV Total pp Cross Section

In this section of BH we have identified some inconsistencies in the evaluation of the quoted quantities, leading to misleading statements and conclusions. As in the last subsection, we shall first present some introductory remarks and then treat each one of the three paragraphs.

• Introductory remarks

The TOTEM result at 7 TeV constitute a high-precision measurement, with quite small uncertainty if compared with any other high-energy experimental result in the accelerator region [14]:

$$\sigma_{tot}^{pp} = 98.3 \pm 0.2^{\text{stat}} \pm 2.8^{\text{syst}} \text{ mb}$$

or, adding the uncertainties in quadrature,

$$\sigma_{tot}^{pp} = 98.3 \pm 2.8 \text{ mb}$$

Therefore, we understand that any acceptable description of this datum demands agreement within the associated uncertainties (datum and prediction, or fit result). Unfortunately, as a first step in the research, we did not evaluate the uncertainty regions in FMS (what has been done in FMS2). However, in this reply, we shall provide additional information on this respect. The error propagation from the fit parameters have been based on the quoted references, namely [8, 15]. The standard error propagation includes variances and covariances associated with all fit parameters [8] and is obtained from the error matrix in the MINUIT code (available to any interested reader by request to Silva at precchia@ifi.unicamp.br).

• First Paragraph - *first two sentences*

- Commentary
Block and Halzen compare our curves in Fig. 5 stating that the result with V4 from Table 3 ($\sqrt{s}_{max}$ = 1.8 TeV) "goes slightly inside the lower error bar of the plotted Totem result."

- Our reply
From our introductory remarks we do not consider this result in agreement with the TOTEM datum. See also what follows.

• First Paragraph - *last two sentences*

- Commentary
Block and Halzen state that, from the V4 parameters in Table 3 ($\sqrt{s}_{max}$ = 1.8 TeV) in a standard error evaluation, they find

$$\sigma_{tot} = \mathbf{96.2} \pm 4.5 \text{ mb}.$$

and conclude that the result "is in excellent agreement with the experimental value of 98.1 ± 2.3 found by Totem."

- Our reply
First, the TOTEM result is 98.3 ± 2.8 mb (as quoted above) and not 98.1 ± 2.3 mb as referred to in BH. Second, from the V4 parameters in Table 3 ($\sqrt{s}_{max}$ = 1.8 TeV) and including the variances and covariances, the correct result is

$$\sigma_{tot} = \mathbf{95.3} \pm 1.7 \text{ mb}, .$$

which does not describe the TOTEM datum adequately. In particular, the above central value does not reach the lower extreme in the TOTEM uncertainty, namely 95.5 mb. Without the covariances we obtain $\sigma_{tot} = 95.3 \pm 5.4$ mb.

The quoted value given by Block and Halzen (96.2 ± 4.5 mb) *does not correspond* to the V4 parameters in Table 3 ($\sqrt{s}_{max}$ = 1.8 TeV), but to the V4 parameters in Table 4 ($\sqrt{s}_{max}$ = 7 TeV). Including the covariances, this value reads 96.2 ± 1.5 mb. Therefore we are faced here with serious errors in the evaluations by Block and Halzen, leading to a misleading interpretation and conclusion.

• Second Paragraph

- Commentary

The reference here is on our DF result in Table 1 ($\sqrt{s}_{max}$ = 1.8 TeV and $\gamma = 2$ fixed). According to their evaluation, at 7 TeV,

$$\sigma_{tot} = 95.4 \pm 3.7 \text{ mb } (1\sigma \text{ diagonal error from } \beta)$$

$$\sigma_{tot} = 95.4 \pm 8.8 \text{ mb } (1\sigma \text{ diagonal error in } s_h \text{ included})$$

leading them to conclude: "... DF model is also in good agreement with the Totem cross section, 98.1 ± 2.3 mb."

- Our reply

Beyond the already mentioned error in the TOTEM result, taking into account the full error matrix (variances and covariances) the correct prediction reads

$$\sigma_{tot} = 95.4 \pm 1.8 \text{ mb}$$

The central value lies below the lower extreme in the TOTEM uncertainty, namely 95.5 mb and the uncertainty region (± 1.8 mb) barely reach the lower error bar. We see no agreement of this result with the TOTEM datum, 98.3 ± 2.8 mb.

• Third Paragraph

- Commentary

"As in the preceding Section, we find that the FMS $\gamma = 2$ fits at low energy, after allowing for errors in the predictions due to the statistical errors in the fitting parameters, successfully predict the Totem total cross section at 7 TeV, thus negating the necessity for considering a violation of the Froissart bound. In simpler words, the FMS fits are consistent with a saturated Froissart bound when the Totem point is included."

- Our reply

First, in our opinion, the uncertainties quoted by Block and Halzen have been estimated in a "rather cumbersome" way (compare the procedures in the previous paragraphs 1 and 2). The standard error propagation [8] include all variances and covariances given by the MINUIT error matrix. Since covariances can assume negative values, the resulting uncertainty may be smaller than the one obtained by taking into account only the variances (diagonal errors). Second, from our arguments on the above paragraphs and those in the last section, we can not agree with the statement "In simpler words, the FMS fits are consistent with a saturated Froissart bound when the Totem point is included."

III. DISCUSSION

In this section we discuss some aspects of the FMS analysis and further information that can be of interest. A short critical commentary on the BH analysis and results is also presented.

A. On the FMS Analysis and Results

The FMS analysis was motivated by the recent theoretical work by Ya. I. Azimov [19] (see also [20, 21]), together with the well known results, obtained a long time ago, by Almaldi *et al.* [7] and the UA4/2 Collaboration [23], which have indicated

$$\gamma = 2.10 \pm 0.10 \quad \text{and} \quad \gamma = 2.25^{+0.35}_{-0.31}$$

respectively. As commented in FMS, Azimov arguments that it is not obvious if Martin's derivation, in the context of axiomatic local quantum field theory, can be directly applied to hadronic processes (QCD). He discusses that "under different assumptions about asymptotic behavior of nonphysical amplitude, the total cross section could grow even faster than $\ln^2 s$" [19]. Moreover, "Increase of the total cross section faster than the log-squared energy does not mean violation of unitarity and is *not forbidden* by any general principles, contrary to a widespread opinion" [20].

These have been the reasons why we have referred to the Froissart-Martin bound, namely the log-squared bound. However, the novel aspect of FMS consisted in *only two phenomenological points*:

1. the first inclusion of the 7 TeV TOTEM result for σ_{tot} in the dataset to be reduced;

2. the investigation of some consequences of this inclusion in what concern a $\ln^{\gamma}[s/s_h]$ leading high-energy contribution.

Our analysis has indicated that, with the Amaldi *et al.* parametrization and the above inclusion, the TOTEM datum cannot be adequately described in the case of $\gamma = 2$ (which corresponds to the COMPETE highest-rank parametrization). The 2012 analysis by the PDG, including the TOTEM point, corroborates this indication [13].

Some explanation on this important point is in order. The *prediction* by the COMPETE Collaboration from the 2002 analysis, with their highest-rank parametrization ($\gamma = 2$) [10, 11], shows agreement with the TOTEM datum at 7 TeV [14]. However, that does not mean the COMPETE highest-rank parametrization can describe this point *once it is included in the dataset*. That is exactly what the 2012 PDG result has indicated, corroborating the conclusion previously presented in FMS. Note that the 2012 PDG edition (Table 46.2 in [13]) quotes both the Azimov article [19] and FMS (the reference, however, corresponds to [22]).

To illustrate the above point, we display in Figure 1 three results obtained through the COMPETE highest-rank parametrization (Eq. (1) with $\gamma = 2$). The curves have been obtained with the values of the parameters from

- the 2002 COMPETE analysis (Table VIII in [11]);

- the 2010 PDG edition (Table 41.2 in [12]), used as input in FMS;

- the 2012 PDG edition [13] (Table 46.2 in [13]), which includes the TOTEM datum in the dataset.

The values of the corresponding parameters, extracted from the above quoted references, are displayed in Table III. The uncertainty region in Fig. 1, in the case of the 2012 PDG result, has been determined through propagation of the errors in the parameters (fourth column in Table III).

TABLE III: Fit results from the quoted references through parametrization (1) with $\gamma = 2$. The parameters a_1, a_2, α and β are in mb, s_l and s_h in GeV^2 and b_1, b_2 are dimensionless.

	COMPETE 2002 [11]	PDG 2010 [12]	PDG 2012 [13]
a_1	42.1 ± 1.3	42.53 ± 1.35	12.72 ± 0.19
b_1	0.467 ± 0.015	0.458 ± 0.017	0.462 ± 0.002
a_2	32.19 ± 0.94	33.34 ± 1.04	7.35 ± 0.08
b_2	0.5398 ± 0.0064	0.545 ± 0.007	0.550 ± 0.005
s_l	1 (fixed)	1 (fixed)	16.21 ± 0.16
α	35.83 ± 0.40	35.35 ± 0.48	34.71 ± 0.15
β	0.3152 ± 0.0095	0.308 ± 0.010	0.265 ± 0.050
s_h	34.0 ± 0.54	28.9 ± 5.4	16.21 ± 0.16

From Figure 1, the *predictions* by the COMPETE 2002 and PDG 2010 are in agreement with the TOTEM result, but not the data reduction from the PDG 2012, which includes this point in the updated dataset: the curve and uncertainty region lie below the TOTEM lower uncertainty bar, *corroborating*, therefore, *the results and conclusions previously presented in FMS*. Note also that these results (Figure 1) corroborate another conclusion in FMS, relating the TOTEM point with the highest cosmic-ray estimates for the pp total cross section (Fly's Eye Collaboration [24] and Pierre Auger Collaboration [25]). In fact, in what concerns our fit results with the Amaldi *et al.* parametrization (and in the particular case of Figure 1 with $\gamma = 2$, namely the COMPETE highest-rank result), there is no agreement among these three points: curves in consistency with the TOTEM datum lie above the central values of the cosmic-ray estimations and the same is true in the inverse sense.

Based on all these facts, we understand that, once included in the dataset, it is, at least, not obvious the 7 TeV TOTEM result can be described by a standard $\ln^2[s/s_h]$ leading dependence and that was the essential "discovery" in FMS. From our analysis, this effect is related with the fundamental

correlation between γ and the scale factor s_h (and also with the subtraction constant K in case of simultaneous fits to σ_{tot} and ρ data: see Appendix A in FMS2 on the correlation matrices). Note, from Table III, that in the COMPETE case, $s_h \sim 34$ GeV^2 and in the PDG 2012 edition, $s_h \sim 16$ GeV^2 (below, therefore, $s_{min} = 25$ GeV^2): the effect of these differences can be seen in Fig. 1. In particular, *once included in the dataset*, it is not expected the 7 TeV TOTEM datum might be described for fixed $\gamma = 2$ and $s_h = 2m_p^2$, as is the case in BH.

The TOTEM Collaboration has already obtained three new high-precision measurements of the total cross section at 7 TeV, through different methods and techniques [26]. All the measurements are consistent within their uncertainties and therefore confirm the first result they have obtained (which has been used in FMS and FMS2). In this respect, we can advance that the inclusion of these three points in our dataset leads also to solutions with γ greater than two [27, 28], corroborating the conclusions in FMS. However, the asymptotic ratio between elastic/total-cross-section, discussed in FMS2, is still under investigation. A luminosity-independent measurement at 8 TeV has been also reported, indicating $\sigma_{tot} = 101.6 \pm 2.9$ mb [29]. This value lies above the prediction in FMS2, namely 98.7 ± 1.0 mb, for $\gamma = 2.346 \pm 0.013$ and $s_h = 0.383 \pm 0.041$ GeV^2 [3].

At this point, we could conjecture (if not speculating) on the implication of a possible increase of σ_{tot} faster than $\ln^2 s$. In contrast with an effective violation of the Froissart-Martin bound, a fast rise of the total cross section might also be associated with some local effect at the LHC energy region, so that, asymptotically, the bound remains valid. A faster-than-squared-logarithm rise points also to the possibility of a power-like behavior [30–32], which has always been and important and representative approach (see, for example, the unitarized model [33, 34]). These conjectures are not in disagreement with the recent theoretical arguments by Azimov [19–21].
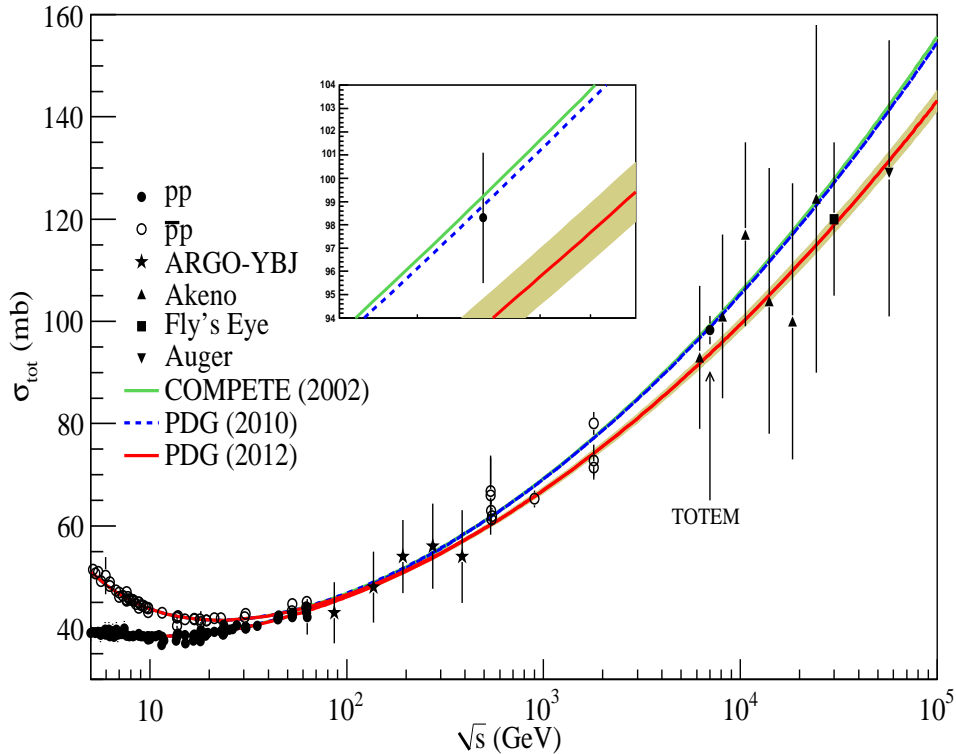

FIG. 1: Results for the pp and $\bar{p}p$ total cross section with the COMPETE highest-rank parametrization ($\gamma = 2$) from the 2002 COMPETE analysis [10, 11], the 2010 PDG version [12] and the 2012 PDG version [13] (which includes the 7 TeV TOTEM datum in the fitted dataset).

B. Brief Commentary on the BH Analysis and Results

Here we present some critical comments on the BH parametrization, the Aspen Model and the BH results for the total, elastic and inelastic cross sections.

1. Analytical Parametrization

The BH analytical parametrization for the total cross sections, used in global fits to σ_{tot} and ρ-values from pp (+) and $\bar{p}p$ (-) scattering, reads [1]

$$\sigma^{\pm}(\nu) = \beta_{P'} \left[\frac{\nu}{m}\right]^{\mu-1} \pm \delta \left[\frac{\nu}{m}\right]^{\alpha-1} + c_0 + c_1 \ln\left(\frac{\nu}{m}\right) + c_2 \ln^2\left(\frac{\nu}{m}\right), \tag{2}$$

where ν is the laboratory energy, m is the proton mass and, in terms of the c.m. energy, $\nu/m \approx s/2m^2$.

Comparison with the parametrization used in FMS, Eq. (1), shows that, for $\gamma = 2$ (fixed), both forms have the same *analytical structure*. However, the striking difference between FMS and BH approaches concerns the number of fixed and free parameters and mainly, in the BH case, the way the parameters are fixed, as discussed in what follows.

In the FMS analysis, except for a dimensional notation choice, namely $s_l = 1$ GeV^2, all the 8 parameters involved have been treated as free or fixed in the 6 variants considered. That is, there is no *ad hoc* fixed parameters, except for particular variant tests. In all cases, the uncertainties in the free parameters have been explicitly given. Moreover, by letting free different parameters in a *nonlinear* fit, we are able to investigate all the correlations involved, the variances, covariances and, as a consequence, the global uncertainties in all fitted and predicted quantities, as done in FMS2.

In the BH approach, besides the arbitrary fixed energy scale (corresponding to $s_h = 2m^2$ at both low and high energy regions), among the 7 parameters in Eq. (2), 5 are fixed (without uncertainties) and only 2 are free in data reductions. Specifically, from Table III in [35]:

- Fixed parameters: $\beta_{P'} = 31.10$ mb, $\mu = 0.5$, $\delta = -28.56$ mb, $\alpha = 0.415$ and $c_0 = 37.32$ mb;

- Fitted parameters: $c_1 = -1.440 \pm 0.070$ mb and $c_2 = 0.2817 \pm 0.0064$ mb.

As a consequence, the parametrization is linear in any reduction to σ_{tot} data, leading to unique solution [8]. That, obviously, contrasts with Eq. (1) and with the strategies in FMS and FMS2, as commented above.

In BH the Reggeons intercepts are fixed, corresponding to $b_1 = 0.5$ and $b_2 = 0.585$ in Eq. (1), which is not in agreement with the spectroscopic data (Chew-Frautschi plots) and scattering fit results, as obtained by several authors [10–13, 36–40]. Since this assumption permeated the intermediate and low energy region, in our opinion, it puts some limits on the reliability of a formal connection with the Finite Energy Sum Rules at low energy. The fixed parameters do not allow the study of correlations and their effects in the fitted and, most importantly, in the predicted quantities. We shall return to this point in the next subsection on the Aspen model.

We also note that, although (1) and (2) have the same analytical structure, the BH and FMS high energy formulations for $\gamma = 2$ are not equivalent, even in the cases (variants) with b_1 and b_2 fixed. In fact, in FMS all the other parameters are free and the fit extends *simultaneously* to both low- and high-energy data (σ_{LE} and σ_{HE} contributions in FMS). As a consequence there is strong correlations among the parameters from both σ_{LE} and σ_{HE} (see, for example, the V4$_1$ coefficients in Table II in the case of a_1 and s_h, α and β). Since that is not the case in BH formulation ($\beta_{P'}$ and δ, corresponding to a_1 and a_2, are fixed), we see no correspondence between the high energy formulations, as stated in BH (after Eq. (7) in that paper).

In the BH approach the main hypothesis concerns the imposing that "the fits to the high energy data smoothly join the cross section and energy dependence obtained by averaging the resonances at low energy" [35]. In the FMS analysis, on the contrary, "We have tried to identify possible high-energy effects that may be unrelated to the trends of the lower-energy data..." [2]. Therefore, the assumptions, approaches and strategies in FMS and BH are completely different. We see no reason for the comparative discussion presented in BH.

2. Aspen Model

In the Aspen model [41, 42] two fundamental quantities, the mass scale m_0 and the coupling constant α_s, are unknown parameters, fixed to *ad hoc* values of 600 MeV and 0.5, respectively in order to obtain best fits in data analysis. The ϵ parameter, from the gluon structure function, is also fixed at 0.05.

The two fundamental parameters, m_0 and α_s, have been reinterpreted by Luna *et al.* in the context of a Dynamical Gluon Mass (DGM) approach [43, 44]. That has allowed connections with nonperturbative QCD, as expected in the soft sector represented by the elastic scattering processes. In the DGM approach, two essential parameters are the dynamical gluon mass scale m_g and the soft Pomeron intercept ϵ. More recently and most important for our purposes, Fagundes, Luna, Menon and Natale (FLMN) have developed a detailed analysis on the influence in the evaluated quantities associated with physical intervals for the m_g and ϵ parameters [45, 46]. Moreover, in a similar way as done recently by Achilli *et al.* [47], FLMN have established bounds and uncertainty regions in all evaluated quantities (fitted and predicted), in accordance with the relevant physical intervals for m_g and ϵ. The main conclusion is that the uncertainty regions play a crucial role in the energy region above that used in the data reductions. In other words, the relevant intervals for each parameters affect substantially the high-energy predictions and they cannot be fixed at *ad hoc* values, without a clear physical justification.

The above conclusion, expressed in [46], puts serious limits on the predictions of phenomenological models constructed on the basis of fixed parameters, whose numerical values do not have an explicit justification and/or whose *consequences in the evaluated quantities are not investigated or even discussed*. In our opinion, these limitations are present in the foundations of both the BH analytical parametrization and the Aspen model.

3. Experimental Evidence

In order to connect their global analytical (empirical) fits to σ_{tot} and ρ data with the inelastic cross section, σ_{inel}, Block and Halzen use the predictions from the Aspen model, in a kind of hybrid approach (semi-empirical or perhaps semi-phenomenological) [1, 4]. The model prediction is parametrized by an analytic expression (Eq. (8) in [1]), with fixed mass and power parameters, without any reference to the uncertainties in the fit parameters. With this hybrid approach, from the model evaluation of σ_{inel} and the fit result for σ_{tot} they infer that

$$\frac{\sigma_{inel}(s)}{\sigma_{tot}(s)} \to 0.509 \pm 0.021 \qquad \text{as} \qquad s \to \infty.$$

a result statistically consistent with the black-disc limit.

As commented in FMS2, we understand that the least ambiguous way to estimate the inelastic cross section is through the s-channel unitarity, $\sigma_{inel} = \sigma_{tot} - \sigma_{el}$, as has been done by the TOTEM Collaboration [14]. That avoids the model dependence involved in direct estimation of σ_{inel}, due to single and double diffraction contributions.

In this respect, it may be interesting to note the prediction in BH for the inelastic cross secion at 7 TeV (Fig. 3 in [1]): the uncertainty region barely reaches the lower error bar in the TOTEM result. Moreover, for the total cross section, the central value of the prediction reaches the low extreme bar of the TOTEM result (and the uncertainty region, above the central value, lies through the lower error bar of the TOTEM point). It should be also noted that as a one-channel eikonal formalism, the Aspen model does not take explicit account of the inelastic diffractive contributions.

In what concerns the BH total cross section prediction at 57 TeV and the recent estimation of this quantity by the Pierre Auger Collaboration [25], we stress a peculiar statement in [1] (also present in [4]):

> "In particular, the agreement with the new highest energy (57 TeV) experimental measurement of both σ_{tot} and σ_{inel} is striking."

First, certainly these are not *experimental measurements*, due to the strong model dependence involved, as already discussed by several authors and also in [25]. They constitute estimations of these quantities because they are based on extrapolations from models tested only in lower energies. Moreover, recall that the estimation for the *pp* total cross section at 57 TeV reads [25]

$$\sigma_{tot}^{pp} = [133 \pm 13 \,(\text{stat})_{-20}^{+17} \,(\text{sys}) \pm 16 \,(\text{Glauber})] \,\text{mb}.$$

As discussed in FMS2 (and also in our section II.B), systematic and, here, theoretical (Glauber) uncertainties are equally probable quantities (do not follow a Gaussian distribution). That means the above central value is equally likely to lie in any place limited by the corresponding nonstatistical uncertainties, namely around $\pm$ 23 mb. Therefore, we see no physical meaning in a statement referring to a "striking agreement".

At last, we understand that the BH analysis and the Aspen model represent important contributions in the investigation of the high-energy elastic hadron scattering. However, given the model character and *ad hoc* assumptions involved we see no conclusive evidence that the BH results constitute an unique and exclusive solution for the rise of the total cross section at high energies. In this respect, a reanalysis by Block and Halzen, including in their dataset the four high-precision TOTEM measurements and presenting their linear fit prediction at higher energies, might be instructive.

IV. CONCLUSIONS AND FINAL REMARKS

In FMS we have presented a study on the rise of the total hadronic cross section, with focus on the recent 7 TeV TOTEM result. The analysis was based on a specific class of analytical parameterization, Eq. (1). Since the effects of all parameters involved have been considered, we were faced with a nonlinear data reduction, which constitutes a non-trivial problem.

Our strategy in FMS (and FMS2) has been to investigate a grid of different physical choices for feedbacks and the corresponding solutions. In both FMS and FMS2, beyond a second distinct possibility, we have considered "conservative" choices for the initial values, namely results previously obtained by the PDG with $\gamma = 2$. The data reductions have been developed through standard running of the MINUIT, namely the default MINUIT error analysis. Perhaps, in the fits presented in FMS a true minimum had not been reached in some cases. However, we understand that this does not invalidate the *general and global conclusions* from our "grid strategy" approach (summarized in section II and, the corresponding results, in Fig. 8 of FMS). Anyway, further analysis, looking for optimizations in the use of the MINUIT code, including also tests with other computational tools for nonlinear fits (as the subroutine MINOS [17]), are certainly important and we intend to implement that.

The lack of unique solutions in our nonlinear fit procedures has been referred to in FMS and FMS2. We have never claimed to have obtained *unique or absolute solutions*. In particular, we have concluded in FMS (the equations refer to that paper),

> "From our data reductions through parametrization (3-5) to pp and $\bar{p}p$ scattering above 5 GeV, including the 7 TeV TOTEM result, we conclude that the total hadronic cross section may rise faster than $\ln^2 s$ at high energies."

> "Our results suggest that the energy dependence of the hadronic total cross section at high energies still constitutes an open problem."

and in FMS2 we have stressed,

> "We also emphasize that our results represent possible consistent statistical solutions for the behavior of the total cross section, but do not correspond to unique solutions."

It is important to note that at the highest energies, once treated as free fit parameters, the exponent γ and the scale factor s_h are correlated in nonlinear data reductions. In the lack of formal and/or theoretical information on the value of this scale factor and without the *canonical assumption* $\gamma = 2$, different solutions, in agreement with the experimental data, can be obtained for the leading $\ln^{\gamma}(s/s_h)$ contribution. Therefore, we understand that the analytical description of the rise of the total hadronic cross section at high energies still constitutes an open problem, demanding further investigations. New high-precision data to be published and to be obtained by the TOTEM Collaboration at 8 - 14 TeV are expected to shed light on the subject.

At last, based on both the "*commentary*" by Block and Halzen and this "*reply*", we see no evidence that invalidate the analysis, results, conclusions and the honest statements expressed in FMS.

Acknowledgments

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